6.3

I Am Having a Series Craving (For Some Math)!

Geometric Series

LEARNING GOALS

In this lesson, you will:

- Generalize patterns to derive the formula for the sum of a finite geometric series.
- Compute a finite geometric series.

KEY TERM

- geometric series

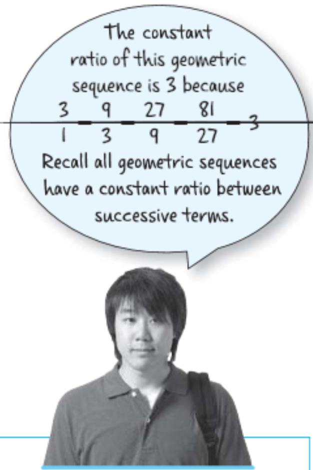
The art that is produced in a culture often reflects the peoples' social values, struggles, and important events over a given time period. While it is generally not considered one of the great art forms of our time, television drama is an art that regularly reflects current events and social issues.

Consider *Mission Impossible*, a spy series which brought millions of viewers the secret assignments of a group of government agents battling dictators around the globe. It's no accident that this series was hugely popular in the 1960's, a time of heightened Cold War anxieties. During the 1980s, a time when more women entered the work force, *Cagney and Lacey* featured a career-focused, single mother battling crime. During the 2000s, *West Wing* focused on political scandals, terrorism, and other foreign affairs issues that were in the news during that period.

What are some of the pressing current events right now? Are they reflected in any popular television series you watch?

PROBLEM 1 Geometric Series Episode 1: The Rise of Euclid

A **geometric series** is the sum of the terms of a geometric sequence. Recall, that the sequence 1, 3, 9, 27, 81 is a geometric sequence because the ratio of any two consecutive terms is constant. Adding the terms creates the geometric series 1 + 3 + 9 + 27 + 81.



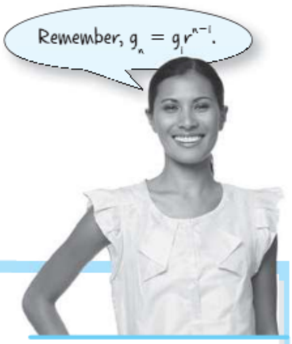
Theresa raises her hand and claims that she has a "trick" for quickly calculating the sum of any geometric series. She asks members of the class to write any geometric series on the board. She boasts that she can quickly tell them how to determine the sum without adding all of the terms. Several examples are shown.

Paul: "OK, so prove it! What is the sum of 1 + 3 + 9 + 27 + 81 + 243 + 729?"	**Theresa:** "Multiply 729(3) and subtract 1. Then divide by 2."
Stella: "What is 5 + 20 + 80 + 320 + 1280 + 5120?"	**Theresa:** "I will have the answer if I multiply 5120(4), subtract 5, and then divide by 3."
Julian: "Let me see . . . How about 10 + 50 + 250 + 1250?"	**Theresa:** "No problem. Multiply 1250(5), subtract 10, and then divide by 4."
Henry: "Hmmm . . . I bet I can stump you with 10 + (−20) + 40 + (−80) + 160."	**Theresa:** "Pretty sneaky with the negatives, Henry, but the method still works. Multiply 160(−2) and subtract 10. This time divide by −3."

1. Verify that Theresa is correct for each series.

2. What is Theresa's "trick"? Describe in words how to calculate the sum of any geometric sequence.

3. Use Theresa's "trick" to calculate $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$. Show all work and explain your reasoning.



Theresa's "trick" really isn't a trick. It is known as Euclid's Method. An example of this method, along with a justification for each step, is shown.

Compute $\sum_{i=1}^{5} 3^{i-1}$.

$S_5 = 1 + 3 + 9 + 27 + 81$	• The common ratio is 3.
$3S_5 = \quad\ \ 3 + 9 + 27 + 81 + 243$ $S_5 = 1 + 3 + 9 + 27 + 81$ (each term $\times 3$)	• Write $3S_n$ above the original series. Multiply each term of the original series by the common ratio. Line up each product above the original series.
$3S_5 = \quad\ \ 3 + 9 + 27 + 81 + 243$ $-S_5 = (1 + 3 + 9 + 27 + 81)$ $2S_5 = -1 + 243$	• Subtract to determine $3S_n - S_n = 2S_n$.
$\frac{2S_5}{2} = \frac{242}{2}$ $S_5 = 121$	• Divide by 2.

In all of the examples, Theresa knew that she could calculate each sum by first multiplying the last term by the common ratio and subtracting the first term. Then she could divide that quantity by one less than the common ratio.

In other words, $S_n = \frac{\text{(Last Term)(Common Ratio)} - \text{(First Term)}}{\text{(Common Ratio} - 1)}$.

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4. Analyze the worked example.

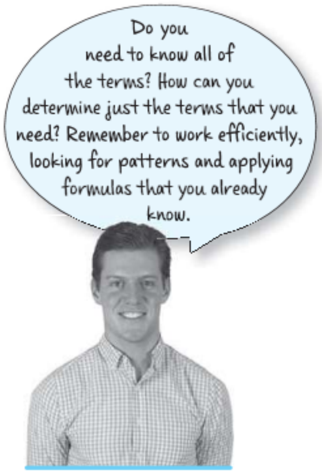
a. In the worked example, why multiply both sides of the equation by 3? Does the algorithm still work if you multiply by a different number? Explain your reasoning.

b. Why do you always divide by one less than the common ratio?

The formula to compute any geometric series becomes $S_n = \frac{g_n(r) - g_1}{r - 1}$, where g_n is the last term, r is the common ratio, and g_1 is the first term.

5. Apply Euclid's Method to compute each.

a. $1 + 10 + 100 + \cdots + 1{,}000{,}000$



b. $10 + 20 + 40 + 80 + 160 + 320$

c. $\sum_{k=1}^{8} 5^{k-1}$

d. A sequence with 9 terms, a common ratio of 2, and a first term of 3.

PROBLEM 2 Return of Long Division: The Pattern Strikes Back

Recall previously you used long division to determine each quotient:

Polynomial Long Division		Rewritten Using the Reflexive and Commutative Properties of Equality
Example 1		
$\frac{r^3 - 1}{r - 1} = r^2 + r + 1$	→	$1 + r + r^2 = \frac{r^3 - 1}{r - 1}$
Example 2		
$\frac{r^4 - 1}{r - 1} = r^3 + r^2 + r + 1$	→	$1 + r + r^2 + r^3 = \frac{r^4 - 1}{r - 1}$
Example 3		
$\frac{r^5 - 1}{r - 1} = r^4 + r^3 + r^2 + r + 1$	→	$1 + r + r^2 + r^3 + r^4 = \frac{r^5 - 1}{r - 1}$

Each Example represents a geometric series, where r is the common ratio and $g_1 = 1$. Each geometric series can be written in summation notation.

Example 1: $n = 3$ $\sum_{i=1}^{3} r^{i-1}$ or $\sum_{i=0}^{2} r^{i}$

Example 2: $n = 4$ $\sum_{i=1}^{4} r^{i-1}$ or $\sum_{i=0}^{3} r^{i}$

1. For each Example, explain why the power of the common ratio in the summation notation is different, yet still represents the series.

2. Identify the number of terms in the series in Example 3, and then write the series in summation notation.

3. Use the pattern generated from repeated polynomial long division to write a formula to compute any geometric series $1 + r + r^2 + r^3 + \cdots + r^{n-1}$ where n is the number of terms in the series, r is the common ratio, and $g_1 = 1$.

$\sum_{i=0}^{n} r^{i} =$ ____________

You can show a proof of $S_n = \frac{r^n - 1}{r - 1}$ where S_n is a series in the form $r^0 + r^1 + r^2 + r^3 + \cdots + r^{n-1}$ with n-terms and a common ratio r.

$$S_n = r^0 + r^1 + r^2 + r^3 + \cdots + r^{n-1}$$

$$rS_n = \quad r^1 + r^2 + r^3 + \cdots + r^{n-1} + r^n$$
$$S_n = r^0 + r^1 + r^2 + \cdots + r^{n-2} + r^{n-1}$$

- Write rS_n above the original series. Multiply each term by r. Line up each product above the original series.

$$rS_n = \quad r^1 + r^2 + r^3 + \cdots + r^{n-1} + r^n$$
$$-\quad S_n = 1 + r^1 + r^2 + r^3 + \cdots + r^{n-1}$$
$$rS_n - S_n = -1 + r^n$$
$$Sn(r - 1) = r^n - 1$$

- Subtract $rS_n - S_n$. Eliminate terms that subtract to 0.

$$\frac{S_n(r - 1)}{(r - 1)} = \frac{(r^n - 1)}{(r - 1)}$$

- Divide by $(r - 1)$.

$$S_n = \frac{r^n - 1}{r - 1}$$

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4. Identify the number of terms, the common ratio, and g_1 for each series. Then compute each.

a. $1 + 2^1 + 2^2 + 2^3 + 2^4$

b. $1 + 5 + 25 + 125 + 625$

c. $1 + (-2) + 4 + (-8) + 16 + (-32)$

5. Angus and Perry each wrote the geometric series 7 + 14 + 28 + 56 + 112 + 224 + 448 + 896 in summation notation and then computed the sum.

Angus

I know that $g_n = g_1r^{n-1}$. The number of terms is 8, the common ratio is 2, and the first term is 7, so I can write the series as $\sum_{i=1}^{8} 7 \cdot 2^{i-1}$.

I know the last term is 896, so I can use Euclid's Method to compute the sum.

$$\frac{896 \cdot 2 - 7}{2 - 1}$$

Perry

I can rewrite the series as $7(1 + 2 + 4 + 8 + 16 + 32 + 64 + 128)$.

I know the common ratio is 2, so I can rewrite the series using powers as $7(2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7)$.

The number of terms is 8, so I can write the series in summation notation as $7\sum_{i=1}^{8} 2^{i-1}$.

Then, I can compute the series as $7\left(\frac{2^8 - 1}{2 - 1}\right)$.

Verify that both methods produce the same sum.

The formula to compute a geometric series that Perry used is $S_n = \frac{g_1(r^n - 1)}{r - 1}$.

Recall Euclid's Method to compute a geometric series is $S_n = \frac{g_n(r) - g_1}{r - 1}$.

You can use the fact that $g_n = g_1r^{n-1}$ to verify that these two formulas are equivalent.

$S_n = \frac{g_n(r) - g_1}{r - 1}$	• Given Euclid's Method.
$= \frac{g_1r^{n-1}(r) - g_1}{r - 1}$	• Substitute $g_n = g_1r^{n-1}$.
$= \frac{g_1r^n - g_1}{r - 1}$	• Perform multiplication.
$= \frac{g_1(r^n - 1)}{r - 1}$	• Factor out g_1.

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6. When is it appropriate to use each formula?

7. Rewrite each series using summation notation.

a. 4 + 12 + 36 + 108 + 324

b. 64 + 32 + 16 + 8 + 4 + 2 + 1

8. Compute each geometric series.

a. $\sum_{i=1}^{4} 6^{i-1}$

b. $10\sum_{i=0}^{4} 3^{i}$

c. $6\sum_{i=0}^{4}\left(\frac{1}{3}\right)^{i}$

9. Analyze the table of values.

x	$f(x)$	$\frac{f(x+1)}{f(x)}$
0	3	
1	4.5	
2	6.75	
3	10.125	
4	15.1875	
5		

a. Complete the table.

b. Describe any patterns that you notice.

c. Assume the geometric sequence continues, determine $f(0) + f(1) + \cdots + f(9)$. Show all work and explain your reasoning

d. Explain why the ratio of any two consecutive terms in a geometric sequence is always a constant.

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PROBLEM 3 Making Choices

1. Jane analyzes the salary schedule for the same position at two different electrical engineering companies, Nothing's Shocking and High Voltage. The salary schedules for the first 5 years are provided with promises from each company that the rate of salary increase will be the same over time.

Time (years)	Nothing's Shocking Salary ($)	High Voltage Salary ($)
1	40,000	46,000
2	42,400	47,840
3	44,944	49,754
4	47,641	51,744
5	50,499	53,813

a. What is the salary in year 10 for each company? Show all work and explain your reasoning.

b. Assuming all other factors are equal, which company offers the better salary over a 10-year period? Show all work and explain your reasoning.

c. What would be the difference in total career salary if you choose one company over the other? Assume a 30-year career. Show all work and explain your reasoning.

2. A single elimination basketball tournament begins with 128 games in the first round. Each round eliminates half of the teams until an overall winner is decided. The tournament sponsor needs to purchase a new ball for every game that will be played throughout the tournament. How many basketballs must the sponsor purchase? Explain your reasoning.

Be prepared to share your solutions and methods.